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Note on Multidimensional Empirical Processes for ϕ-Mixing Random Vectors

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In 1974, Sen proved weak convergence of the empirical processes (in the J_1-topology on $D^p[0, 1]$) for a stationary ϕ-mixing sequence of stochastic $p(\geqslant 1)$-vectors. In this note, we show that Sen's theorem on weak convergence of the multidimensional empirical process for a stationary ϕ-mixing sequence of stochastic vectors remains true under a less restrictive condition on the mixing constants $\{\phi_n\}$, i.e., $\phi_n = O(n^{-1-\delta})$ for some $\delta > 0$.

1. INTRODUCTION

Let $\{\mathbf{X}_i\} = \{(X_{i1}, ..., X_{ip})', -\infty < i < \infty\}$ be a strictly stationary sequence of stochastic $p(\geqslant 1)$-vectors defined on a probability space $(\Omega, \mathscr{A}, P)$ with each X_{ij} $(-\infty < i < \infty, 1 \leqslant j \leqslant p)$ having the uniform distribution on the interval $[0, 1]$. Let $\mathscr{M}_a{}^b$ denote the σ-algebra generated by $\mathbf{X}_i$, $a \leqslant i \leqslant b$. Suppose that the sequence is ϕ-mixing, i.e., the sequence satisfies the condition

$$| P(B \mid \mathscr{M}^0_{-\infty}) - P(B)| \leqslant \phi(n) \downarrow 0 \qquad (n \to \infty) \tag{1.1}$$

for all $B \in \mathscr{M}_n{}^\infty$ with probability one.

Let

$$F_{[j]}(t) = P(X_{ij} \leqslant t) = t, \qquad 0 \leqslant t \leqslant 1, \quad j = 1, ..., p, \tag{1.2}$$

and put

$$F(\mathbf{t}) = P(\mathbf{X}_i \leqslant \mathbf{t}), \qquad \mathbf{t} \in E^p, \tag{1.3}$$

where $E^p = \{t : \mathbf{0} \leqslant \mathbf{t} \leqslant \mathbf{1}\}$ is the p-dimensional unit cube, $\mathbf{0} = (0, ..., 0)$, $\mathbf{1} = (1, ..., 1)$ and $\mathbf{a} \leqslant \mathbf{b}$ means that $a_j \leqslant b_j$, $1 \leqslant j \leqslant p$. Note that $F(\mathbf{t}) = 0$

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if at least one coordinate of $\mathbf{t}$ is 0. For a sample $\mathbf{X}_1, ..., \mathbf{X}_n$ of size n, the empirical df is defined by

$$F_n(\mathbf{t}) = n^{-1} \sum_{i=1}^{n} c(\mathbf{t} - \mathbf{X}_i), \qquad \mathbf{t} \in E^p, \qquad n \geqslant 1, \tag{1.4}$$

where $c(\mathbf{u}) = 1$ if and only if $\mathbf{u} \geqslant \mathbf{0}$, and 0 otherwise. $F_n(\mathbf{t}) = 0$, when at least one coordinate of $\mathbf{t}$ is zero. Let $W_n = \{W_n(\mathbf{t}) : \mathbf{t} \in E^p\}$ be the empirical processes defined by

$$W_n(\mathbf{t}) = n^{1/2}[F_n(\mathbf{t}) - F(\mathbf{t})], \qquad \mathbf{t} \in E^p, \quad n \geqslant 1. \tag{1.5}$$

For every $n \geqslant 1$, the process W_n belongs to the space $D^p[0, 1]$ of all real-valued functions on E^p with no discontinuities of the second kind, and with $D^p[0, 1]$ we associate the (extended) Skorokhod J_1-topology. Let $W = \{W(\mathbf{t}) : \mathbf{t} \in E^p\}$ be the p-dimensional Gaussian process where

$$EW(\mathbf{t}) = 0, \qquad \mathbf{t} \in E^p \tag{1.6}$$

and for every $\mathbf{s}, \mathbf{t} \in E^p$

$$\begin{aligned} R(\mathbf{s}, \mathbf{t}) &= EW(\mathbf{s})\, W(\mathbf{t}) \\ &= E\{c(\mathbf{s} - \mathbf{X}_1)\, c(\mathbf{t} - \mathbf{X}_1) - F(\mathbf{s}) F(\mathbf{t})\} \\ &\quad + \sum_{k=2}^{\infty} E\{c(\mathbf{s} - \mathbf{X}_1)\, c(\mathbf{t} - \mathbf{X}_k) + c(\mathbf{s} - \mathbf{X}_k)\, c(\mathbf{t} - \mathbf{X}_1) - 2F(\mathbf{s}) F(\mathbf{t})\} \end{aligned} \tag{1.7}$$

(It is known that the series on the right side of (1.7) converges if $\sum \phi(n) < \infty$ holds.)

For $p = 1$, Billingsley [1] proved first that the weak convergence of $\mathscr{W}_n$ to $\mathscr{W}$ holds under the condition $\sum n^2 \phi^{1/2}(n) < \infty$ and the author [5] proved the same result under the condition $\phi(n) = O(n^{-2})$ using Sen's method in [3]. Further, for general $p \geqslant 1$, Sen [4] proved that the weak convergence of $\mathscr{W}_n$ to $\mathscr{W}$ holds under the condition $\sum n\phi^{1/2}(n) < \infty$.

The object of this note is to show that the above results remain true under the less restrictive condition $\phi(n) = O(n^{-1-\delta})$ for some δ (>0), i.e., to show the following

THEOREM. *Suppose that $\{\mathbf{X}_i\}$ is a strictly stationary ϕ-mixing sequence of stochastic vectors, defined in Section* 1. *Then W_n converges in law (in the Skorokhod J_1-topology on $D^p[0, 1]$) to the Gaussian process W, defined above, if $\phi(n) = O(n^{-1-\delta})$ for some $\delta > 0$.*

2. Proof

First, we consider a basic lemma. Let the process $\{z_i\}$ be a strictly stationary ϕ-mixing sequence of Bernoullian random variables, centered at expectations. Let $\alpha = Ez_1^2 > 0$. Then, $E\mid z_1 \mid \leqslant \alpha^{1/2}$. Further, let $S_n = z_1 + \cdots + z_n$.

Lemma. *If $\phi(n) = O(n^{-1-\delta})$ for some δ $(0 < \delta < 1)$, then for any $\epsilon > 0$ and for all n sufficiently large there exists a τ (>0) such that*

$$P(\mid n^{-1/2}S_n \mid \geqslant \epsilon) \leqslant K_1\{n^{-\tau}\alpha + \alpha^{1+(\delta/2)}\}, \tag{2.8}$$

where the constant K_1 does not depend on n and α.

Proof. In what follows, by the letter K_i, we denote any positive quantity (not always the same) which is bounded and does not depend on n and α. Let n be sufficiently large. Let r be the largest integer such that $2^{r+1} \leqslant n$. Put $p = 2^{[r\beta]}(\beta = (8 - 3\delta)/16)$ and $m = 2^{r-[r\beta]}$, where $[s]$ denotes the largest j such that $j \leqslant s$. Moreover, let $\xi_j = \sum_{i=1}^{p} z_{j+i}$ $(j = 1,..., 2m)$ and put

$$T(k) = T_k = \sum_{j=1}^{k} \xi_{2j-2}, \qquad T'_m = \xi_{2j-1}, \qquad T''_m = S_n - T_m - T'_m .$$

Then, using Lemma 1 in [1, p. 170], $\mid z_i \mid \leqslant 1$ and $E\xi_0 = 0$, we have the following:

(i) $\mid T''_m \mid \leqslant \sum_{i=2mp+1}^{n} \mid z_i \mid \leqslant n - 2mp \leqslant 2p = o(n^{1/2})$.

(ii) $E\xi_0^2 \leqslant K_2 p\alpha$ and $|E\xi_0\xi_{2j} \mid \leqslant 2E\xi_0^2\{\phi(jp)\}^{1/2} \leqslant K_3 p^{(1-\delta)/2}j^{-(1+\delta)/2}\alpha$,

and so

$$ET_k^2 = E\left|\sum_{j=1}^{k} \xi_j\right|^2 \leqslant K_4 kp\alpha\{1 + p^{-(1+\delta)/2}k^{(1-\delta)/2}\} \leqslant K_5 kp\alpha \qquad (1 \leqslant k \leqslant m) \tag{2.9}$$

and

$$E\mid T_k \mid^{2+\delta} \leqslant \{kp\}^{\delta}ET_k^2 \leqslant K_6(kp)^{1+\delta}\alpha \qquad (1 \leqslant k \leqslant m). \tag{2.10}$$

(iii) If for some $\gamma'(>0)$

$$\{\phi(p)\}^{1/(2+\delta)} < \gamma'/8, \tag{2.11}$$

then

$$E\left|\sum_{i=1}^{2k} \xi_i\right|^{2+\delta} \leqslant (2 + \gamma)\, E\left|\sum_{i=1}^{k} \xi_i\right|^{2+\delta} + 4\left\{E\left|\sum_{i=1}^{n} \xi_i\right|^{2}\right\}^{1+(\delta/2)} \tag{2.12}$$

(cf. [2, Lemma 18.5.1]).

Now, choose two positive numbers γ and ρ for which

$$(2+\gamma)\,2^{-(1+\delta^2/(4+\delta))} < \rho < 1 \tag{2.13}$$

holds. Since (2.11) holds for all n sufficiently large, so using (12) repeatedly it follows from (2.9) and (2.10) that

$$\begin{aligned} E\mid T_m\mid^{2+\delta} &\leqslant (2+\gamma)^{r-[r\beta]-[\delta r/16]} E\mid T(2^{[\delta r/16]})\mid^{2+\delta} \\ &\quad + 4 \sum_{i=1}^{r-[r\beta]-[\delta r/16]} (2+\gamma)^{i-1}\{E\mid T(2^{r-[r\beta]-i})\mid^2\}^{1+(\delta/2)} \\ &\leqslant K_7(2+\gamma)^{r-[r\beta]-[\delta r/16]}\{p2^{[\delta r/16]}\}^{1+\delta}\alpha \\ &\quad + 4K_8 \sum_{i=1}^{r-[r\beta]-[\delta r/16]} (2+\gamma)^{i-1}\{p2^{r-[r\beta]-i}\alpha\}^{1+(\delta/2)} \end{aligned} \tag{2.14}$$

and consequently

$$\begin{aligned} 2^{-r(1+(\delta/2))}E\mid T_m\mid^{2+\delta} &\leqslant K_9\rho^{r-[r\beta]-[r\delta/16]}\alpha \\ &\quad + K_{10} \sum_{i=1}^{r-[r\beta]-[\delta r/16]} \left(\frac{2+\gamma}{2^{1+(\delta/2)}}\right)^{i+1} \alpha^{1+(\delta/2)}. \end{aligned} \tag{2.15}$$

It follows from (2.13) and (2.15) that there exists a ρ' $(0<\rho'<1)$ such that the left-hand side of (2.15)

$$\leqslant K_9(\rho')^r\alpha + K_{11}\alpha^{1+(\delta/2)}. \tag{2.16}$$

On the other hand, from (i)

$$\begin{aligned} P(\mid S_n\mid > 3\epsilon n^{1/2}) &\leqslant 2P(\mid T_m\mid > \epsilon n^{1/2}) + P(\mid T_m''\mid > \epsilon n^{1/2}) \\ &= 2P(\mid T_m\mid > \epsilon n^{1/2}) \\ &\leqslant K_{11}n^{-(1+(\delta/2))}E\mid T_m\mid^{2+\delta}. \end{aligned} \tag{2.17}$$

Hence, from (2.16) and (2.17), we have the desired result.

Now, the proof of Theorem follows along the same line as of the proof of Theorem 2.1 in Sen [4] using our Lemma instead of using Lemma 2.1 in Sen [3] and so is omitted.

Remark. It is obvious that we can also prove Theorem 2.2 in Sen [4] under the condition $\phi(n) = O(n^{-1-\delta})$.

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